NYO-1480-102
MF-57

Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

Stability of Dissipative Systems

E. M. Barston

AEC Research and Development Report

Physics
July 15, 1968

New York University

UNCLASSIFIED

New York University
Courant Institute of Mathematical Sciences
Magneto-Fluid Dynamics Division

MF-57 NYO-1480-102

STABILITY OF DISSIPATIVE SYSTEMS

E. M. Barston

15 July 1968

Submitted for publication to
Archive for Rational Mechanics and Analysis.

U.S. Atomic Energy Commission
Contract No. AT(30-1)-1480

UNCLASSIFIED

Abstract

The stability of a class of "smooth" solutions $\xi(t)$ to an equation of the form $P\ddot{\xi} + K\dot{\xi} + H\xi(t) = 0$ is discussed in terms of $\|\xi(t)\|$. P, K, and H are time-independent linear formally self-adjoint operators defined in an inner-product space, and $P \geq 0$, $K \geq 0$. Necessary and sufficient conditions for exponential stability are given in terms of an energy principle, and the maximal growth rate Ω of an unstable system is shown to be the supremum of a certain functional over the class of "negative energy" states. Sufficient conditions for the attainment of Ω (i.e., that Ω lie in the point spectrum) are given.

Table of Contents

I. Introduction

The equations of small oscillations about a state of equilibrium of a system subject to dissipative as well as conservative forces often assumes the form[1-5,7]

$$P\ddot{\xi} + K\dot{\xi} + H\xi(t) = 0 , \qquad t \geq 0 \tag{1}$$

where P, K, and H are time-independent linear formally self-adjoint operators in an inner product space E, with $P \geq 0$ and $K \geq 0$. The operator K represents the dissipative forces, H the conservative forces. The linear stability of such equilibria is determined by the boundedness of the solutions of Eq. (1) for arbitrary allowed initial conditions; the equilibrium is said to be stable if all the solutions of Eq. (1) are bounded independently of t, and unstable otherwise.

Kelvin and Tait[2] proposed a simple necessary and sufficient condition for exponential stability for real operators $P > 0$, K, and H on a finite-dimensional Euclidean space E. The system described by Eq. (1) is exponentially stable if and only if the system in the absence of dissipative forces (i.e., Eq. (1) with $K \equiv 0$) is exponentially stable, or in other words, every solution $\xi(t)$ of Eq. (1) satisfies $\|\xi(t)\| \leq Me^{\epsilon t}$, $t \geq 0$, for every $\epsilon > 0$ and some

constant $M(\epsilon)$ if and only if $\inf_E \frac{(\zeta,H\zeta)}{(\zeta,\zeta)} \geq 0$. (Kelvin and Tait did not prove their assertion; a proof using the methods of Liapunov can be found in Ref. 3). Exponential stability of the system for $H \geq 0$ is a simple consequence of the fact that the energy of the system, given by $(\dot{\xi},P\dot{\xi}) + (\xi,H\xi)$, is a nonincreasing function of t for $K \geq 0$ (see Theorem I of Sec. II).

Exponential instability for $\inf_E \frac{(\zeta,H\zeta)}{(\zeta,\zeta)} < 0$ can be guaranteed under far more general conditions. Indeed, the following result is an immediate consequence of Theorem V of Ref. 6.

Theorem: Let P, K, and H be linear Hermitian operators on and into the Hilbert space E, K and H be completely continuous, $P > 0$ and invertible (i.e., $\inf_E \frac{(\zeta,P\zeta)}{(\zeta,\zeta)} > 0$). Let $\inf_E \frac{(\zeta,H\zeta)}{(\zeta,\zeta)} < 0$. Then H has n negative eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n < 0$ where $n \geq 1$, and there exists n positive real numbers $\omega_1 \geq \omega_2 \geq \dots \geq \omega_n > 0$ and nonzero vectors $\zeta_1, \zeta_2, \dots, \zeta_n \varepsilon E$ such that $\xi_\ell(t) \equiv e^{\omega_\ell t}\zeta_\ell$ satisfies Eq. (1) for $\ell = 1,2,\dots,n$ and $(\zeta_k,\zeta_\ell) = 0$ if $\omega_k = \omega_\ell$.

We consider a much larger class of problems in Sec. II. There it is assumed that P, K, and H are merely formally self-adjoint operators on their domains of definition D_P, D_K, and D_H, which are subsets of some inner product space E, and that $P \geq 0$, $K \geq 0$, and H is bounded below. (We say that an operator L is formally self-adjoint if $(\eta,L\zeta) = (L\eta,\zeta)$

for all $\eta,\zeta\varepsilon D_L$.) Stability is discussed in terms of the norm of solutions of Eq. (1) belonging to a certain "smooth" class S_o. No spectral analysis is made; we operate directly with the time-dependent equation. The basic idea involved is very simple, if we assume for the moment that everything is sufficiently "nice", as it is if E is finite-dimensional. If $\inf_E \frac{(\zeta,H\zeta)}{(\zeta,\zeta)} \geq 0$, it is easily shown that all the "smooth" solutions of Eq. (1) are exponentially bounded in norm. If $\inf_E \frac{(\zeta,H\zeta)}{(\zeta,\zeta)} < 0$, it is not difficult to show that Eq. (1) admits of a solution $\xi(t)$ satisfying $\|\xi(t)\| \geq \delta > 0$ for some positive δ. Then we merely observe that $\zeta(t) = e^{-\omega t}\xi(t)$ satisfies

$$P\ddot{\zeta} + K_\omega\dot{\zeta} + H_\omega\zeta(t) = 0 \ , \quad t \geq 0 \ , \tag{2}$$

if and only if $\xi(t)$ satisfies Eq. (1), where $K_\omega \equiv 2\omega P+K \geq 0$ for $\omega \geq 0$, $H_\omega \equiv \omega^2 P + \omega K + H$ and K_ω are both formally self-adjoint, so that Eq. (2) is of the same type as Eq. (1). Then for every positive ω for which $\inf_E \frac{(\zeta,H_\omega\zeta)}{(\zeta,\zeta)} < 0$, there is a $\zeta(t)$ satisfying Eq. (2) such that $\|\zeta(t)\| \geq \delta > 0$ for $t \geq 0$. Hence $\xi(t) = e^{\omega t}\zeta(t)$ satisfies Eq. (1), and

$$\|\xi(t)\| \geq \delta e^{\omega t} \ , \quad t \geq 0 \ . \tag{3}$$

The maximal growth rate Ω of the system is then obtained as the supremum of the set of all ω's for which

$\inf\limits_{E} \frac{(\zeta,H_\omega\zeta)}{(\zeta,\zeta)} < 0$. This is the essence of the program carried out in Sec. II. In order to facilitate the computation of Ω, we show that it can also be characterized as the supremum of the functional Ω_η (defined in Sec. II) over the set of vectors η for which $(\eta,H\eta) < 0$. Applications to specific problems will be considered in another paper.

II. Stability Theorems

Let E be a linear inner product space with inner product (,) and P, K, and H linear formally self adjoint operators (independent of the parameter t) with domains D_P, D_K, and D_H in E. For $-\infty < \omega < \infty$ we define $K_\omega \equiv 2\omega P + K$, $H_\omega \equiv \omega^2 P + \omega K + H$, and the set S_ω is the set of all vector functions $\xi(t)$ of the parameter t defined for all $t \geq 0$ satisfying the following nine conditions:

1. $\xi(t)\varepsilon D_P \cap D_{K_\omega} \cap D_{H_\omega} \; (= D_P \cap D_K \cap D_H)\,, \quad t \geq 0$ (4)

2. $\dot{\xi}(t)\varepsilon D_P \cap D_{K_\omega} \; (= D_P \cap D_K)\,, \quad t \geq 0$ (5)

3. $\ddot{\xi}(t)\varepsilon D_P\,, \quad t \geq 0$ (6)

4. $P\ddot{\xi} + K_\omega\dot{\xi} + H_\omega\xi(t) = 0\,, \quad t \geq 0$ (7)

5. $\frac{d}{dt}(\dot{\xi},P\dot{\xi}) = (\ddot{\xi},P\dot{\xi}) + (\dot{\xi},P\ddot{\xi}) \quad t \geq 0$ (8)

6. $\frac{d}{dt}(\dot{\xi},P\xi) = (\ddot{\xi},P\xi) + (\dot{\xi},P\dot{\xi}) \quad t \geq 0$ (9)

7. $\frac{d}{dt}(\xi,P\xi) = (\dot{\xi},P\xi) + (\xi,P\dot{\xi}) \qquad t \geq 0$ (10)

8. $\frac{d}{dt}(\xi,K_\omega\xi) = (\dot{\xi},K_\omega\xi) + (\xi,K_\omega\dot{\xi}) \qquad t \geq 0$ (11)

9. $\frac{d}{dt}(\xi,H_\omega\xi) = (\dot{\xi},H_\omega\xi) + (H_\omega\xi,\dot{\xi}) \qquad t \geq 0$ (12)

Note: The precise definition of the t-derivative $\dot{\xi}$ is not important in the sequel, provided that the usual rules for differentiating sums and products (of scalars and vectors) are valid. Thus one can think of $\dot{\xi}$ as being defined in the norm topology of E, or, if E is an n-fold Cartesian product of L_2-spaces (as is often the case in physical applications), $\dot{\xi}$ can be taken to be the n-vector obtained by computing the partial derivative with respect to t of each of the n components of $\xi(t)$.

It is clear that S_ω is homogeneous (i.e., $\xi(t) \varepsilon S_\omega$ implies $\alpha\xi(t) \varepsilon S_\omega$ for all real numbers α) and translation invariant (i.e., $\xi(t) \varepsilon S_\omega$ implies $\xi(t+T) \varepsilon S_\omega$ for each fixed $T \geq 0$). We also have

<u>Lemma I</u>: Let $\omega \varepsilon (-\infty,\infty)$. Then $S_\omega = e^{-\omega t}S_0$, i.e., $\zeta(t) \varepsilon S_\omega$ if and only if $\zeta(t) = e^{-\omega t}\xi(t)$ for some $\xi(t) \varepsilon S_0$.

<u>Proof</u>: The lemma follows directly from the formulas

$\frac{d}{dt}[e^{\omega t}\xi(t)] = e^{\omega t}[\dot{\xi}+\omega\xi]$ and $\frac{d^2}{dt^2}[e^{\omega t}\xi(t)] = e^{\omega t}[\ddot{\xi}+2\omega\dot{\xi}+\omega^2\xi]$.

The stability theorems to follow will refer to solutions of Eq. (1) in the class S_0, which may, in virtue of the defining Eqs. (4)-(12) be thought of as the class of "suitably smooth" solutions of Eq. (1). Eqs. (5)-(12) are merely the usual rules for differentiating inner products; Eqs. (4) and (5) offer no restriction on the solutions of Eq. (1) provided $D_P \supset D_K \supset D_H$, but become additional "smoothness" requirements should the above set relation not hold.

We now introduce a number of definitions. Let $D \equiv D_P \cap D_K \cap D_H$. The set $\{\eta | \eta = \xi(0),\ \xi(t) \varepsilon S_\omega\}$, defined for each fixed real ω, is independent of ω by Lemma I. Denote this set by Y. Y is homogeneous, and for each $\xi(t) \varepsilon S_\omega$, $\xi(T) \varepsilon Y$ for every $T \geq 0$. We shall use the letter Q to denote any homogeneous subset of Y. The set $\{\eta | \eta = \xi(T),\ T \geq 0,\ \xi(t) \varepsilon S_\omega \text{ and } \xi(0) \varepsilon Q\}$, defined for each fixed real ω, is independent of ω by the homogeneity of Q and Lemma I. Denote this set by Q^*. Then Q^* is homogeneous and $Y \supset Q^* \supset Q$. For any $S \subset D$ we define $F_S(\omega) \equiv \inf_S \dfrac{(\zeta, H_\omega \zeta)}{(\zeta,\zeta)}$ for $\omega \varepsilon (-\infty, \infty)$. Let Z denote the set of all ordered pairs $\langle \xi(0), \dot{\xi}(0) \rangle$ for $\xi(t) \varepsilon S_0$. We define B to be the class of all homogeneous subsets Q of Y with the property that for every $\eta \varepsilon Q$, and each real α, there exists ϕ_α such that $\langle \eta, \phi_\alpha \rangle \varepsilon Z$ and $\phi_\alpha - \alpha\eta \varepsilon N$, where N is the nullspace of P. If $Q \varepsilon B$, we say that Q is basic.

Lemma II: A) Let $\xi(t)\varepsilon S_\omega$ for some real ω. Then

$$\frac{d}{dt}\{(\dot{\xi},P\dot{\xi})+(\xi,H_\omega\xi)\} = -2(\dot{\xi},K_\omega\dot{\xi}), \quad t \geq 0 \tag{13}$$

If in addition, $P \geq 0$ and $K_\omega \geq 0$ on $D_P \cap D_K$, $\xi(0)\varepsilon Q$, and $F_{Q^*}(\omega) > -\infty$, we have

$$F_{Q^*}(\omega)\ \|\xi(t)\|^2 \leq (\dot{\xi}_o,P\dot{\xi}_o)+(\xi_o,H_\omega\xi_o), \quad t \geq 0 \tag{14}$$

B) Let Q be basic, $F_Q(\omega) < 0$, $F_{Q^*}(\omega) > -\infty$, $P \geq 0$ and $K_\omega \geq 0$ on $D_P \cap D_K$. Then there exists $\zeta(t)\varepsilon S_o$ and a constant $\delta > 0$ such that $\dot{\zeta}(0)-\omega\zeta(0)\,\varepsilon\, N$ and $\|\zeta(t)\| \geq \delta e^{\omega t}$ for all $t \geq 0$.

Proof: Eq. (13) follows at once from Eqs. (7), (8), and (12). Eq. (13) and $K_\omega \geq 0$ imply that $E(t) \equiv (\dot{\xi},P\dot{\xi})+(\xi,H_\omega\xi)$ is a nonincreasing function of t for $t \geq 0$, so that

$$(\xi,H_\omega\xi) \leq E(0) - (\dot{\xi},P\dot{\xi}) \leq E(0), \quad t \geq 0 \tag{15}$$

for $P \geq 0$. Suppose $\xi(0)\varepsilon Q$. Then by the definition of Q^*, we have, for each $\xi = \xi(T)$ with $\|\xi\| > 0$,

$$F_{Q^*}(\omega) \equiv \inf_{Q^*} \frac{(\zeta,H_\omega\zeta)}{(\zeta,\zeta)} \leq \frac{(\xi,H_\omega\xi)}{(\xi,\xi)} \leq \frac{E(0)}{\|\xi\|^2} \tag{16}$$

Note that $E(0) < 0$ implies $\|\xi(t)\| > 0$ for all $t \geq 0$ by Eq. (15), so that Eq. (16) yields Eq. (14). Now suppose

$Q\varepsilon B$, $F_Q(\omega) < 0$, $P \geq 0$ and $K \geq 0$ on $D_P \cap D_K$. Since $Q \subset Q^*$, $F_{Q*}(\omega) \leq F_Q(\omega) < 0$. Now $F_Q(\omega) < 0$ implies the existence of an $\eta\varepsilon Q$ for which $(\eta, H_\omega \eta) < 0$. Since Q is basic, there exists $\zeta(t)\varepsilon S_o$ such that $\zeta(0) = \eta$, $\dot{\zeta}(0) - \omega\eta \ \varepsilon \ N$. Then $\xi(t) \equiv e^{-\omega t}\zeta(t)\varepsilon S_\omega$, $\xi(0) = \eta$, $\dot{\xi}(0) = \dot{\zeta}(0) - \omega\zeta(0) \ \varepsilon \ N$, so that Eq. (14) yields

$$\|\zeta(t)\| = \|\xi(t)\| e^{\omega t} \geq \delta \ e^{\omega t} \ , \quad t \geq 0$$

where $\delta \equiv \{(\eta, H_\omega \eta)/F_{Q*}(\omega)\}^{\frac{1}{2}} > 0$.

This completes the proof of Lemma II.

We shall assume throughout the remainder of this section that $K \geq 0$ and $P \geq 0$ on $D_P \cap D_K$.

Let $S \subset D$. We introduce the following definitions:

$$V_S \equiv \{\omega | F_S(\omega) < 0 \ , \quad -\infty < \omega < \infty\}$$

$$\Omega(S) \equiv \begin{cases} -\infty & V_S \text{ empty} \\ \sup\limits_{V_S} \omega & V_S \text{ nonempty} \end{cases}$$

$$\tilde{S} \equiv \{\eta | \eta\varepsilon S, \quad (\eta, H\eta) < 0\}$$

For each $\eta\varepsilon\tilde{D}$, we define the positive functional

$$\Omega_\eta \equiv \begin{cases} \frac{1}{2}\left\{\left[\frac{(\eta,K\eta)^2}{(\eta,P\eta)^2} - 4\,\frac{(\eta,H\eta)}{(\eta,P\eta)}\right]^{\frac{1}{2}} - \frac{(\eta,K\eta)}{(\eta,P\eta)}\right\} & (\eta,P\eta) > 0 \\ -\frac{(\eta,H\eta)}{(\eta,K\eta)} & (\eta,P\eta) = 0,\ (\eta,K\eta) > 0 \\ \infty & (\eta,P\eta) = 0 = (\eta,K\eta) \end{cases}$$

The next lemma shows that $\Omega(S)$ is positive and equals the supremum of the functional Ω_η over $\tilde{S}$, provided that $F_S(0) < 0$ and that $P \geq 0$ and $K \geq 0$ on S.

Lemma III: A) Let $S \subset D$, $K \geq 0$ and $P \geq 0$ on S. Then $F_S(\omega)$ is a nondecreasing function of ω on $[0,\infty)$. If in addition, H is bounded below on S and $\inf_S \frac{(\eta,[K+\alpha P]\eta)}{(\eta,\eta)} > 0$ for all $\alpha > 0$, then $F_S(\omega)$ is strictly increasing on $[0,\infty)$.
B) Let $S \subset D$, $K \geq 0$ and $P \geq 0$ on S, and $F_S(0) < 0$. Then $\tilde{S}$ is nonempty, for each $\eta \varepsilon \tilde{S}$ we have $F_S(\omega) < 0$ for all $\omega \varepsilon [0,\Omega_\eta)$, and $\Omega(S) = \sup_{\eta \varepsilon \tilde{S}} \Omega_\eta > 0$. (Thus $K \geq 0$ and $P \geq 0$ on S and $\Omega(S) \leq 0$ imply $F_S(0) \geq 0$.)
C) Let $S \subset D$, $\Omega(S) > 0$, and $F_S(\omega)$ be strictly increasing on $[0,\infty)$. Then $F_S(\omega) > 0$ for $\omega > \Omega(S)$ and $F_S(\omega) < 0$ for $0 \leq \omega < \Omega(S)$.
Proof: A) Let $\omega \varepsilon [0,\infty)$ and $\epsilon > 0$. Then

$$F_S(\omega+\epsilon) = \inf_S \frac{(\eta,H_{\omega+\epsilon}\eta)}{(\eta,\eta)} = \inf_S \left\{\frac{(\eta,H_\omega\eta)}{(\eta,\eta)} + \epsilon\,\frac{(\eta,[K+(2\omega+\epsilon)P]\eta)}{(\eta,\eta)}\right\}$$

$$\geq F_S(\omega) + \epsilon \inf_S \frac{(\eta,[K+(2\omega+\epsilon)P]\eta)}{(\eta,\eta)} \tag{17}$$

which proves A). Note that $P \geq 0$ and $K \geq 0$ on S and $F_S(0) = \inf_S \frac{(\eta, H\eta)}{(\eta,\eta)} > -\infty$ imply $F_S(\omega) > -\infty$ for $\omega \geq 0$.

B) $F_S(0) < 0$ means $\tilde{S}$ is nonempty. For each $\eta \varepsilon \tilde{S}$, we have

$$F_S(\omega) = \inf_S \frac{(\zeta, H_\omega \zeta)}{(\zeta,\zeta)} \leq G_\eta(\omega)$$

where

$$G_\eta(\omega) \equiv \frac{(\eta, H_\omega \eta)}{(\eta,\eta)} = ||\eta||^{-2} \{(\eta, H\eta) + \omega(\eta, K\eta) + \omega^2(\eta, P\eta)\}$$

If $\Omega_\eta = \infty$, then $G_\eta(\omega) < 0$ for all $\omega \varepsilon [0,\infty)$. If $\Omega_\eta < \infty$, then $G_\eta(\omega)$ is a strictly increasing function of ω for $\omega \varepsilon [0,\infty)$, and $G_\eta(\Omega_\eta) = 0$. Thus, in any case, $F_S(\omega) \leq G_\eta(\omega) < 0$ for $\omega \varepsilon [0, \Omega_\eta)$. This implies $\Omega_\eta \leq \Omega(S)$ for every $\eta \varepsilon \tilde{S}$, so that $\sup_{\eta \varepsilon \tilde{S}} \Omega_\eta \leq \Omega(S)$. We now show that $\Omega(S) \leq \sup_{\eta \varepsilon \tilde{S}} \Omega_\eta$. Let $0 < \omega < \Omega(S)$. Then $F_S(\omega) < 0$, for $F_S(\omega)$ is nondecreasing on $[0,\infty)$ by Lemma III A), so that $F_S(\omega) \geq 0$ would imply $F_S(\lambda) \geq F_S(\omega) \geq 0$ for all $\lambda \geq \omega$, which contradicts the definition of $\Omega(S)$. $F_S(\omega) < 0$ means that there exists $\eta \varepsilon \tilde{S}$ such that $G_\eta(\omega) < 0$. Now $G_\eta(\lambda) \geq 0$ for $\lambda \geq \Omega_\eta$, and therefore $\omega < \Omega_\eta$. Hence $\omega < \sup_{\eta \varepsilon \tilde{S}} \Omega_\eta$ for all $\omega \varepsilon (0, \Omega[S])$, which implies $\Omega(S) \leq \sup_{\eta \varepsilon \tilde{S}} \Omega_\eta$. This proves B).

C). Let $\omega = \Omega(S) + \epsilon$, $\epsilon > 0$. The definition of $\Omega(S)$ implies that $F_S(\lambda) \geq 0$ for all $\lambda \geq \Omega(S)$. Suppose $F_S(\omega) = 0$. Then since $F_S(\omega)$ is strictly increasing, $F_S(\omega - \frac{\epsilon}{2}) < 0$, which is a contradiction. Now suppose $0 \leq \omega < \Omega(S)$. Then $F_S(\omega) < 0$, for $F_S(\omega) \geq 0$ and F_S nondecreasing would imply $F_S(\lambda) \geq 0$ for all $\lambda \geq \omega$, which contradicts the definition of $\Omega(S)$.

<u>Theorem I</u>: Let $P \geq 0$ and $K \geq 0$ on $D_P \cap D_K$.

A) If $F_D(0) > 0$, then for every $\xi(t)\varepsilon S_o$ we have

$$\|\xi(t)\| \leq \left\{\frac{(\dot{\xi}_0, P\dot{\xi}_0) + (\xi_0, H\xi_0)}{F_D(0)}\right\}^{\frac{1}{2}}, \quad t \geq 0 \tag{18}$$

B) If $F_D(0) = 0$ and $\Delta \equiv \inf\limits_{D_P \cap D_K} \frac{(\zeta, P\zeta)}{(\zeta,\zeta)} > 0$, then for every $\xi(t)\varepsilon S_o$ for which $\frac{d}{dt}\|\xi\|^2 = (\dot{\xi},\xi) + (\xi,\dot{\xi})$ $(t \geq 0)$ we have

$$\|\xi(t)\| \leq \left\{\frac{(\dot{\xi}_0, P\dot{\xi}_0) + (\xi_0, H\xi_0)}{\Delta}\right\}^{\frac{1}{2}} t + \|\xi_0\|, \quad t \geq 0 \tag{19}$$

C) If $F_D(0) = 0$ and $F_D(\omega) > 0$ for $\omega > 0$, then for every $\xi(t)\varepsilon S_0$ and every positive ϵ we have

$$\|\xi(t)\| \leq \left\{\frac{(\dot{\zeta}_0, P\dot{\zeta}_0) + (\xi_0, H_\epsilon\xi_0)}{F_D(\epsilon)}\right\}^{\frac{1}{2}} e^{\epsilon t}, \quad t \geq 0\,. \tag{20}$$

where $\dot{\zeta}_0 = \dot{\xi}_0 - \epsilon\xi_0$.

Proof: A) For any Q, $Q \subset Q^* \subset D$, so that $0 < F_D(0) \leq F_{Q*}(0)$, and Eq. (18) follows at once from Eq. (14) of Lemma II.

B) Let $\xi(t) \varepsilon S_o$. $F_D(0) = 0$ implies $(\xi, H\xi) \geq 0$ for all $t \geq 0$, and Eq. (15) of Lemma II gives

$$\Delta\|\dot{\xi}\|^2 \leq (\dot{\xi}, P\dot{\xi}) + (\xi, H\xi) \leq E_o \ , \quad t \geq 0 \tag{21}$$

so that $\|\dot{\xi}(t)\| \leq (E_o/\Delta)^{\frac{1}{2}}$ for all $t \geq 0$. Now $2\|\xi\| \dfrac{d\|\xi\|}{dt} = \dfrac{d\|\xi\|^2}{dt}$ $= (\dot{\xi}, \xi) + (\xi, \dot{\xi}) \leq 2\|\dot{\xi}\|\|\xi\|$ for $\|\xi\| > 0$, so that $\dfrac{d\|\xi\|}{dt} \leq \|\dot{\xi}\| \leq (E_o/\Delta)^{\frac{1}{2}}$ for $\|\xi(t)\| > 0$. It follows easily from the mean value theorem that

$$\|\xi(t)\| \leq (E_o/\Delta)^{\frac{1}{2}}t + \|\xi_0\| \ , \quad t \geq 0$$

which is just Eq. (19).

C) Clearly $F_D(\epsilon) > 0$. Let $\xi(t)\varepsilon S_o$. Then $\zeta(t) \equiv e^{-\epsilon t}\xi(t)\varepsilon S_\epsilon$, and Eq. (14) of Lemma II gives

$$\|\xi(t)\| = e^{\epsilon t} \ \|\zeta(t)\| \leq e^{\epsilon t} \left\{ \frac{(\dot{\zeta}_0, P\dot{\zeta}_0) + (\zeta_0, H_\epsilon \zeta_0)}{F_D(\epsilon)} \right\}^{\frac{1}{2}}$$

which is Eq. (20).

Theorem II: A) Let $K \geq 0$ and $P \geq 0$ on $D_P \cap D_K$, Q be basic, $F_Q(0) < 0$, and $F_D(0) > -\infty$. Then $\Omega(Q) > 0$, and for every $\omega\varepsilon[0, \Omega(Q))$ there exists $\zeta(t)\varepsilon S_o$ and a constant $\delta > 0$ such that $\dot{\zeta}(0) - \omega\zeta(0) \varepsilon N$ and $\|\zeta(t)\| \geq \delta e^{\omega t}$ for all $t \geq 0$.

B) Let $K \geq 0$ and $P \geq 0$ on $D_P \cap D_K$, $F_{Q*}(0) < 0$, and $F_{Q*}(\omega)$ be strictly increasing for $\omega > \Omega(Q^*)$. Then for every $\zeta(t) \varepsilon S_o$ with $\zeta(0) \varepsilon Q$ and each $\epsilon > 0$ there exists a constant $\rho > 0$ such that $\|\zeta(t)\| \leq \rho e^{[\Omega(Q^*)+\epsilon]t}$, $t \geq 0$.

Proof: A) $\Omega(Q) > 0$ by Lemma III-B. $F_Q(\omega)$ is nondecreasing on $[0,\infty)$ by Lemma III-A, so that $F_Q(\omega) < 0$ for $0 \leq \omega < \Omega(Q)$. $F_D(0) > -\infty$ implies $F_{Q*}(\omega) > -\infty$ for $\omega \geq 0$, and the result now follows at once from Lemma II-B.

B) $\Omega(Q^*) > 0$ by Lemma III-B. For $\epsilon > 0$, $F_{Q*}[\Omega(Q^*)+\epsilon] > 0$ since $F_{Q*}(\omega)$ is strictly increasing on $(\Omega(Q^*),\infty)$. Let $\zeta(t) \varepsilon S_o$ and $\zeta(0) \varepsilon Q$. Then $\xi(t) \equiv e^{-[\Omega(Q^*)+\epsilon]t}\, \zeta(t) \;\varepsilon\; S_{\Omega+\epsilon}$, and Eq. (14) of Lemma II yields

$$\|\zeta(t)\| = \|\xi(t)\| e^{[\Omega+\epsilon]t} \leq \rho e^{[\Omega+\epsilon]t} \,, \quad t \geq 0$$

where

$$\rho^2 \equiv \frac{(\dot{\xi}_0, P\dot{\xi}_0) + (\xi_0, H_{\Omega+\epsilon}\xi_0)}{F_{Q*}(\Omega+\epsilon)} > 0 \,.$$

Theorem III: Let $-\infty < F_D(0) < 0$ and $\inf\limits_{D_P \cap D_K} \dfrac{(\eta,[K+\alpha P]\eta)}{(\eta,\eta)} > 0$ for $\alpha > 0$. Suppose there is a basic Q for which $\Omega(Q) = \Omega(D)$. Then the system described by Eq. (1) is exponentially unstable with maximal growth rate $\Omega(D)$, i.e., for each $\omega \varepsilon [0,\Omega(D))$ there exists $\zeta(t) \varepsilon S_o$ and a constant $\delta > 0$ such that $\|\zeta(t)\| \geq \delta\, e^{\omega t}$ for all $t \geq 0$, and for every $\xi(t) \varepsilon S_o$ and

every $\epsilon > 0$ there exists a constant $\rho > 0$ such that $\|\xi(t)\| \leq \rho\, e^{[\Omega(D)+\epsilon]t}$ for all $t \geq 0$.

Proof: Note that $D \subset D_P \cap D_K$ and that $\inf_{D_P \cap D_K} \frac{(\eta,[K+\alpha P]\eta)}{(\eta,\eta)} > 0$ for $\alpha > 0$ implies that $K \geq 0$ and $P \geq 0$ on $D_P \cap D_K$. $\Omega(D) > 0$ by Lemma III-B; $F_D(\omega)$, $F_Y(\omega)$ and $F_Q(\omega)$ are strictly increasing on $[0,\infty)$ by Lemma III-A. Since $D \supset Y \supset Q$, $F_D(\omega) \leq F_Y(\omega) \leq F_Q(\omega)$ for all real ω, and therefore $\Omega(D) = \Omega(Q)$ implies $\Omega(D) = \Omega(Y)$. The theorem is now an immediate consequence of Theorem II (substitute Y for Q in Theorem II-B, and note that $Y^* = Y$).

Theorem IV: Let P, H, and K be (bounded) Hermitian operators on and into the Hilbert space E, with $K \geq 0$ and $\inf_E \frac{(\zeta,P\zeta)}{(\zeta,\zeta)} > 0$ Then $Z = E \times E$, and for

A) $F_E(0) > 0$, Eq. (18) holds for every $\xi(t) \varepsilon S_o$;

B) $F_E(0) = 0$, Eq. (19) holds for every $\xi(t) \varepsilon S_o$;

C) $F_E(0) < 0$, the set of solutions S_o of Eq. (1) is unstable with maximal growth rate $\Omega(E)$.

(Note: The t-derivative $\dot{\xi}(t) = \frac{d\xi(t)}{dt}$ is to be understood as being defined in the norm topology).

Proof: We have $D = D_P = D_H = D_K = E$. If $\eta(t)$ and $\xi(t) \varepsilon E$ for $t \geq 0$ are differentiable (in the norm topology), then for any bounded operator L on E we have $\frac{d}{dt}(\xi,L\eta) = (\dot{\xi},L\eta)+(\xi,L\dot{\eta})$. Thus Eqs. (8) - (12) hold for every $\xi(t)$ which is twice differentiable for $t \geq 0$, and we also have $\frac{d}{dt}(\xi,\xi) = (\dot{\xi},\xi) + (\xi,\dot{\xi})$.

Statements A) and B) follow from Theorem I-A) and B).

$$\text{Let } \zeta_0 = \begin{pmatrix} \zeta_{10} \\ \zeta_{20} \end{pmatrix} \varepsilon E \times E \text{ and define } \zeta(t) = \begin{pmatrix} \zeta_1(t) \\ \zeta_2(t) \end{pmatrix} \equiv e^{At}\zeta_0$$

for $t \geq 0$, where the bounded linear operator A on $E \times E$ is given by $A \equiv \begin{pmatrix} 0 & I \\ -P^{-1}H & -P^{-1}K \end{pmatrix}$. Then $\zeta(t)$ is differentiable (infinitely often) in the norm topology of $E \times E$ and satisfies

$$\dot{\zeta}(t) = \begin{pmatrix} \dot{\zeta}_1(t) \\ \dot{\zeta}_2(t) \end{pmatrix} = A\zeta(t) \qquad \text{for } t \geq 0 .$$

Therefore $\zeta_1(t)$ satisfies Eq. (1) for $t \geq 0$, and since $\zeta(0) = \zeta_0$, $\zeta_1(0) = \zeta_{10}$ and $\dot{\zeta}_1(0) = \zeta_{20}$. But ζ_0 is an arbitrary element of $E \times E$ and $\zeta_1(t) \varepsilon S_0$, so that $Z = E \times E$. Statement (C) follows at once from Theorem III by taking E as the basic Q.

Corollary: Let P, K, and H be linear Hermitian operators on and into the finite-dimensional Euclidean space E, with $P > 0$ and $K \geq 0$. Then the system described by Eq. (1) is exponentially unstable if and only if $F_E(0) < 0$, and the maximum growth rate of the system is given by $\Omega(E)$. (The following theorem shows that $\Omega(E)$ is actually attained.) The system is stable if $F_E(0) > 0$.

Proof: In a finite-dimensional E differentiability in the norm topology and component-wise differentiability are equivalent, as are norm stability and component-wise stability.

Furthermore, uniqueness of solutions is well-known.

The following theorem gives sufficient conditions for the attainment of the maximal growth rate.

Theorem V: Let P, K, and H be (bounded) Hermitian operators on and into the Hilbert space E, having the following properties:

1) $\inf\limits_{E} \frac{(\zeta,[K+\alpha P]\zeta)}{(\zeta,\zeta)} > 0$ for $\alpha > 0$

2) $H_\omega = P_\omega - C_\omega$ for each $\omega > 0$, where P_ω and C_ω are Hermitian operators on and into E, $\inf\limits_{E} \frac{(\zeta,P_\omega\zeta)}{(\zeta,\zeta)} > 0$, and C_ω is completely continuous.

Then if $F_E(0) < 0$, there exists $\eta\varepsilon E$ with $\|\eta\| > 0$ such that $\xi(t) \equiv e^{\Omega(E)t}\eta$ for $t \geq 0$ satisfies Eq. (1).

Proof: It follows easily from the definition of $F_E(\omega)$ and the boundedness of P, K, and H that $F_E(\omega)$ is a continuous function of ω on $[0,\infty)$, and we have $F_E(\omega) \longrightarrow \infty$ as $\omega \longrightarrow \infty$. Then we conclude from Lemma III that $\Omega(E)$ is the unique root of $F_E(\omega)$ in $[0,\infty)$. Therefore

$$0 = F_E(\Omega) \equiv \inf_{E} \frac{(\zeta,H_\Omega\zeta)}{(\zeta,\zeta)} = \inf_{E} \left\{ \frac{(\zeta,P_\Omega\zeta)}{(\zeta,\zeta)} \left[1 - \frac{(\zeta,C_\Omega\zeta)}{(\zeta,P_\Omega\zeta)}\right] \right\} \tag{22}$$

which holds if and only if $1 = \sup\limits_{E} \dfrac{(\zeta, C_\Omega \zeta)}{(\zeta, P_\Omega \zeta)}$, since $\inf\limits_{E} \dfrac{(\zeta, P_\Omega \zeta)}{(\zeta, \zeta)} > 0$. It follows from well-known theorems on completely continuous Hermitian operators that there exists $\eta \varepsilon E$, $\|\eta\| > 0$, such that $P_\Omega \eta = C_\Omega \eta$, i.e., $H_\Omega \eta = 0$. This is clearly the desired η.

Acknowledgement

The author would like to thank Professors H. Weitzner and H. Grad for reading the manuscript. The work presented here was supported by the Magneto-Fluid Dynamics Division, Courant Institute of Mathematical Sciences, New York University, under contract AT(30-1)-1480 with the U. S. Atomic Energy Commission.

References

1. J.W. Strutt (Lord Rayleigh), Theory of Sound (Dover Publications, Inc., New York, 1945) 2nd edition, Chaps. 4,5.

2. W. Thompson (Lord Kelvin) and P.G. Tait, Treatise on Natural Philosophy, Part I (Cambridge University Press, London, 1903) Secs. 339-345.

3. N.G. Chetaev, The Stability of Motion (Pergamon Press, London, 1961) Ch. 5.

4. S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability (Oxford University Press, Oxford, 1961) Ch. 10.

5. P. Lancaster, Lambda-Matrices and Vibrating Systems (Pergamon Press, London, 1966) Ch. 7.

6. E.M. Barston, J. Math. Phys. 8, 523 (1967).

7. E.M. Barston, "Stability of the Resistive Sheet Pinch", Phys. Fluids (To be published).